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Two-parameter deformed multimode oscillators and *q*-symmetric states

W-S Chung

Department of Physics and Research Institute of Natural Science, Gyeongsang National University, Jinju, 660-701, Korea

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Abstract. Two types of coherent states for the two-parameter deformed multimode oscillator system are investigated. Moreover, two-parameter deformed $gl(n)$ algebra and deformed symmetric states are constructed.

1. Introduction

Quantum groups or the q-deformed Lie algebra imply some specific deformations of classical Lie algebras. From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of the Heisenberg algebra was developed by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently, there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6–10].

Recently Greenberg [11] has studied the following q-deformation of the multimode bosonic algebra:

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}$$

where the deformation parameter q has to be real. The main problem of Greenberg's approach is that one cannot derive the relation between the operators a_i at all. In order to resolve this problem, Mishra and Rajasekaran [12] generalized the algebra to the complex parameter q with $|q| = 1$ and another real deformation parameter p. In this paper we use the result of [12] to construct two types of coherent states and q-symmetric states.

2. Two-parameter deformed multimode oscillators

2.1. Representation and coherent states

In this subsection we discuss the algebra given in [12] and develop its representation. Mishra and Rajasekaran's algebra for multimode oscillators is given by

$$\begin{aligned} &a_i a_j^\dagger = q a_j^\dagger a_i \qquad (i < j) \\ &a_i a_i^\dagger - p a_i^\dagger a_i = 1 \\ &a_i a_j = q^{-1} a_j a_i \qquad (i < j) \end{aligned} \tag{1}$$

where $i, j = 1, 2, \ldots, n$. In this case we can say that $a_i^\dagger$ is a Hermitian adjoint of a_i.

The Fock space representation of the algebra (1) can be easily constructed by introducing the Hermitian number operators $\{N_1, N_2, \ldots, N_n\}$ obeying

$$[N_i, a_j] = -\delta_{ij} a_j \qquad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger \qquad (i, j = 1, 2, \ldots, n). \tag{2}$$

From the second relation of (1) and equation (2), the relation between the number operator and the creation and annihilation operators is given by

$$a_i^\dagger a_i = [N_i] = \frac{p^{N_i} - 1}{p - 1} \tag{3}$$

or

$$N_i = \sum_{k=1}^{\infty} \frac{(1-p)^k}{1-p^k} (a_i^\dagger)^k a_i^k. \tag{4}$$

Let $|0, 0, \ldots, 0\rangle$ be the unique ground state of this system satisfying

$$N_i |0, 0, \ldots, 0\rangle = 0 \qquad a_i |0, 0, \ldots, 0\rangle = 0 \qquad (i, j = 1, 2, \ldots, n) \tag{5}$$

and $\{|n_1, n_2, \ldots, n_n\rangle | n_i = 0, 1, 2, \ldots\}$ be the complete set of the orthonormal number eigenstates obeying

$$N_i |n_1, n_2, \ldots, n_n\rangle = n_i |n_1, n_2, \ldots, n_n\rangle \tag{6}$$

and

$$\langle n_1, \ldots, n_n | n_1', \ldots, n_n'\rangle = \delta_{n_1 n_1'} \cdots \delta_{n_n n_n'}. \tag{7}$$

If we set

$$a_i |n_1, n_2, \ldots, n_n\rangle = f_i(n_1, \ldots, n_n) |n_1, \ldots, n_i - 1, \ \ldots, n_n\rangle \tag{8}$$

from the fact that $a_i^\dagger$ is a Hermitian adjoint of a_i, we have

$$a_i^\dagger |n_1, n_2, \ldots, n_n\rangle = f^*(n_1, \ldots, n_i + 1, \ldots, n_n) |n_1, \ldots, n_i + 1, \ \ldots, n_n\rangle. \tag{9}$$

Making use of the relation $a_i a_{i+1} = q^{-1} a_{i+1} a_i$ we find the following relation for the f_i:

$$q \frac{f_{i+1}(n_1, \ldots, n_n)}{f_{i+1}(n_1, \ldots, n_i - 1, \ldots, n_n} = \frac{f_i(n_1, \ldots, n_n)}{f_i(n_1, \ldots, n_{i+1} - 1, \ldots, n_n)}$$

$$|f_i(n_1, \ldots, n_i + 1, \ldots, n_n)|^2 - p|f_i(n_1, \ldots, n_n)|^2 = 1. \tag{10}$$

Solving the above equations we find

$$f_i(n_1, \ldots, n_n) = q^{\sum_{k=i+1}^{n} n_k} \sqrt{[n_i]} \tag{11}$$

where $[x]$ is defined as

$$[x] = \frac{p^x - 1}{p - 1}.$$

Thus the representation of this algebra becomes

$$\begin{aligned} a_i |n_1, \ldots, n_n\rangle &= q^{\sum_{k=i+1}^{n} n_k} \sqrt{[n_i]} |n_1, \ldots, n_i - 1, \ldots, n_n\rangle \\ a_i^\dagger |n_1, \ldots, n_n\rangle &= q^{-\sum_{k=i+1}^{n} n_k} \sqrt{[n_i + 1]} |n_1, \ldots, n_i + 1, \ldots, n_n\rangle. \end{aligned} \tag{12}$$

The general eigenstates $|n_1, n_2, \ldots, n_n\rangle$ are obtained by applying the operators $a_i^\dagger$ to the ground state $|0, 0, \ldots, 0\rangle$:

$$|n_1, n_2, \ldots, n_n\rangle = \frac{(a_n^\dagger)^{n_n} \cdots (a_1^\dagger)^{n_1}}{\sqrt{[n_n]! \cdots [n_1]!}} |0, 0, \ldots, 0\rangle \tag{13}$$

where

$$[n]! = [n][n-1]\cdots[2][1] \qquad [0]! = 1.$$

The coherent states for the $gl_q(n)$ algebra are usually defined as

$$a_i|z_1,\ldots,z_i,\ldots,z_n\rangle_- = z_i|z_1,\ldots,z_i,\ldots,z_n\rangle_- . \tag{14}$$

From the $gl_q(n)$-covariant oscillator algebra we obtain the following commutation relation between the z_i and the z_i^*, where z_i^* is a complex conjugate of z_i:

$$\begin{aligned} z_i z_j &= q z_j z_i \qquad (i<j) \\ z_i^* z_j^* &= \frac{1}{q} z_j^* z_i \qquad (i<j) \\ z_i^* z_j &= q z_j z_i^* \qquad (i \neq j) \\ z_i^* z_i &= z_i z_i^*. \end{aligned} \tag{15}$$

Using these relations the coherent states become

$$|z_1,\ldots,z_n\rangle_- = c(z_1,\ldots,z_n)\sum_{n_1,\ldots,n_n=0}^{\infty}\frac{z_n^{n_n}\cdots z_1^{n_1}}{\sqrt{[n_1]!\cdots[n_n]!}}|n_1,n_2,\ldots,n_n\rangle. \tag{16}$$

Using (13) we can rewrite equation (16) as

$$|z_1,\ldots,z_n\rangle_- = c(z_1,\ldots,z_n)e_p(z_n a_n^\dagger)\cdots e_p(z_1 a_1^\dagger)|0,0,\ldots,0\rangle \tag{17}$$

where

$$e_p(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]!}$$

is a deformed exponential function.

In order to obtain the normalized coherent states, we should impose the condition $\langle z_1,\ldots,z_n|z_1,\ldots,z_n\rangle_- = 1$. Then the normalized coherent states are given by

$$|z_1,\ldots,z_n\rangle_- = \frac{1}{\sqrt{e_p(|z_1|^2)\cdots e_p(|z_n|^2)}}e_p(z_n a_n^\dagger)\cdots e_p(z_1 a_1^\dagger)|0,0,\ldots,0\rangle \tag{18}$$

where $|z_i|^2 = z_i z_i^* = z_i^* z_i$.

2.2. Positive energy coherent states

The purpose of this subsection is to obtain another type of coherent states for the algebra (1). In order to do so, it is convenient to introduce n sub-Hamiltonians as follows

$$H_i = a_i^\dagger a_i - \nu$$

where

$$\nu = \frac{1}{1-p}.$$

Then the commutation relation between the sub-Hamiltonians and mode operators are given by

$$H_i a_j^\dagger = (\delta_{ij}(p-1)+1)a_j^\dagger H_i \qquad [H_i, H_j] = 0. \tag{19}$$

The action of the sub-Hamiltonian on the number eigenstates gives

$$H_i|n_1,n_2,\ldots,n_n\rangle = -\frac{p^{n_i}}{1-p}|n_1,n_2,\ldots,n_n\rangle. \tag{20}$$

Thus the energy becomes negative when $0 < p < 1$. As was noticed in [13], for the positive energy states it is not the a_i but the $a_i^\dagger$ that play the part of of the lowering operator:

$$
\begin{aligned}
&H_i|\lambda_1 p^{n_1},\ldots,\lambda_n p^{n_n}\rangle = \lambda_i p^{n_i}|\lambda_1 p^{n_1},\ldots,\lambda_n p^{n_n}\rangle \\
&a_i^\dagger|\lambda_1 p^{n_1},\ldots,\lambda_n p^{n_n}\rangle = q^{-\sum_{k=i+1}^n n_k}\sqrt{\lambda_i p^{n_i+1}+\nu}|\lambda_1 p^{n_1},\ldots,\lambda_i p^{n_i+1},\ldots,\lambda_n p^{n_n}\rangle \\
&a_i|\lambda_1 p^{n_1},\ldots,\lambda_n p^{n_n}\rangle = q^{\sum_{k=i+1}^n n_k}\sqrt{\lambda_i p^{n_i}+\nu}|\lambda_1 p^{n_1},\ldots,\lambda_i p^{n_i-1},\ldots,\lambda_n p^{n_n}\rangle
\end{aligned}
\tag{21}
$$

where $\lambda_1,\ldots,\lambda_n > 0$.

Due to this fact, it is natural to define coherent states corresponding to the representation (21) as the eigenstates of the $a_i^\dagger$:

$$
a_i^\dagger|z_1,\ldots,z_n\rangle_+ = z_i|z_1,\ldots,z_n\rangle_+. \tag{22}
$$

Because the representation (21) depends on n free parameters λ_i, the coherent states $|z_1,\ldots,z_n\rangle_+$ can take different forms.

If we assume that the positive energy states are normalizable, i.e.

$$
\langle\lambda_1 p^{n_1},\ldots,\lambda_n p^{n_n}|\lambda_1 p^{n_1'},\ldots,\lambda_n p^{n_n'}\rangle = \delta_{n_1 n_1'}\cdots\delta_{n_n n_n'}
$$

and form exactly one series for some fixed λ_i's, we can then obtain

$$
\begin{aligned}
|z_1,\ldots,z_n\rangle_+ = C\sum_{n_1,\ldots,n_n=-\infty}^{\infty}&\left[\prod_{k=0}^{n}\frac{p^{n_k(n_k-1)/4}}{\sqrt{(-\nu/\lambda_k;\,p)_{n_k}}}\left(\frac{1}{\sqrt{\lambda_k}}\right)^{n_k}\right] \\
&\times z_n^{n_n}\cdots z_1^{n_1}|\lambda_1 p^{-n_1},\ldots,\lambda_n p^{-n_n}\rangle.
\end{aligned}
\tag{23}
$$

If we require that ${}_+\langle z_1,\ldots,z_n|z_1,\ldots,z_n\rangle_+ = 1$, we have

$$
C^{-2} = \prod_{k=1}^{n} {}_0\psi_1\left(-\frac{\nu}{\lambda_k};\,p,-\frac{|z_k|^2}{\lambda_k}\right) \tag{24}
$$

where the bilateral p-hypergeometric series ${}_0\psi_1(a;\,p,x)$ is defined [14] by

$$
{}_0\psi_1(a;\,p,x) = \sum_{n=-\infty}^{\infty}\frac{(-)^n p^{n(n-1)/2}}{(a;\,p)_n}x^n. \tag{25}
$$

2.3. Two-parameter deformed $gl(n)$ algebra

The purpose of this subsection is to derive the deformed $gl(n)$ algebra from the deformed multimode oscillator algebra. The multimode oscillators given in (1) can be arrayed in bilinear form to construct the generators

$$
E_{ij} = a_i^\dagger a_j. \tag{26}
$$

From the fact that $a_i^\dagger$ is a Hermitian adjoint of a_i, we know that

$$
E_{ij}^\dagger = E_{ji}. \tag{27}
$$

Then the deformed $gl(n)$ algebra is obtained from the algebra (1):

$$\begin{aligned}
&[E_{ii}, E_{jj}] = 0\\
&[E_{ii}, E_{jk}] = 0 \qquad (i \neq j \neq k)\\
&[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \qquad (i \neq j)\\
&E_{ii}E_{ij} - pE_{ij}E_{ii} = E_{ij} \qquad (i \neq j)\\
&E_{ij}E_{ik} = \begin{cases} q^{-1}E_{ik}E_{ij} & \text{if } j < k \\ qE_{ik}E_{ij} & \text{if } j > k \end{cases}\\
&E_{ij}E_{kl} = q^{2(R(i,k)+R(j,l)-R(j,k)-R(i,l))}E_{kl}E_{ij} \qquad (i \neq j \neq k \neq l)
\end{aligned} \tag{28}$$

where the symbol $R(i, j)$ is defined by

$$R(i, j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i \leqslant j. \end{cases}$$

This algebra goes to an ordinary $gl(n)$ algebra when the deformation parameters q and p go to 1.

3. *q*-symmetric states

In this section we study the statistics of many particle state. Let N be the number of particles. Then the N-partcle state can be obtained from the tensor product of the single-particle states:

$$|i_1, \ldots, i_N\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle \tag{29}$$

where $i_1, \ldots, i_N$ take one value among $\{1, 2, \ldots, n\}$ and the single-particle state is defined by $|i_k\rangle = a^\dagger_{i_k}|0\rangle$.

Consider the case where k appears n_k times in the set $\{i_1, \ldots, i_N\}$. Then we have

$$n_1 + n_2 + \cdots + n_n = \sum_{k=1}^{n} n_k = N. \tag{30}$$

Using these facts we can define the q-symmetric states as follows:

$$|i_1, \ldots, i_N\rangle_q = \sqrt{\frac{[n_1]_{p^2}! \cdots [n_n]_{p^2}!}{[N]_{p^2}!}} \sum_{\sigma \in \mathrm{Perm}} \mathrm{sgn}_q(\sigma)|i_{\sigma(1)} \cdots i_{\sigma(N)}\rangle \tag{31}$$

where

$$\begin{aligned}
&\mathrm{sgn}_q(\sigma) = q^{R(i_1 \cdots i_N)} p^{R(\sigma(1) \cdots \sigma(N))}\\
&R(i_1, \ldots, i_N) = \sum_{k=1}^{N} \sum_{l=k+1}^{N} R(i_k, i_l)
\end{aligned} \tag{32}$$

and $[x]_{p^2} = (p^{2x} - 1)/(p^2 - 1)$. Then the q-symmetric states obey

$$|\ldots, i_k, i_{k+1}, \ldots\rangle_q = \begin{cases} q^{-1}|\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if } i_k \langle i_{k+1} \\ |\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if } i_k = i_{k+1} \\ q|\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if } i_k > i_{k+1}. \end{cases} \tag{33}$$

The above property can be rewritten by introducing the deformed transition operator $P_{k,k+1}$ obeying

$$P_{k,k+1}|\ldots, i_k, i_{k+1}, \ldots\rangle_q = |\ldots, i_{k+1}, i_k, \ldots\rangle_q. \tag{34}$$

This operator satisfies

$$P_{k+1,k}P_{k,k+1} = Id \qquad \text{so} \quad P_{k+1,k} = P^{-1}_{k,k+1}. \tag{35}$$

Then equation (33) can be written as

$$P_{k,k+1}|\ldots, i_k, i_{k+1}, \ldots\rangle_q = q^{-\epsilon(i_k,i_{k+1})}|\ldots, i_{k+1}, i_k, \ldots\rangle_q \tag{36}$$

where $\epsilon(i, j)$ is defined as

$$\epsilon(i, j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j. \end{cases}$$

It is worth noting that the relation (36) does not contain the deformation parameter p. Also, relation (36) goes to the symmetric relation for the ordinary bosons when the deformation parameter q goes to 1. If we define the fundamental q-symmetric state $|q\rangle$ as

$$|q\rangle = |i_1, i_2, \ldots, i_N\rangle_q$$

with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_N$, we have for any k

$$|P_{k,k+1}|q\rangle|^2 = ||q\rangle|^2 = 1.$$

In deriving the above relation we used the following identity:

$$\sum_{\sigma\in\mathrm{Perm}} p^{R(\sigma(1),\ldots,\sigma(N))} = \frac{[N]_{p^2}!}{[n_1]_{p^2}!\cdots[n_n]_{p^2}!}.$$

4. Concluding remarks

In conclusion, we have used the two-parameter deformed multimode oscillator system given in [12] to construct its representation, coherent states and deformed $gl_q(n)$ algebra. The mutimode oscillator is important for investigating many-body quantum mechanics and statistical mechanics. In order to construct the new statistical behaviour for deformed particles obeying the algebra (1), we have investigated the defomed symmetric property of two-parameter deformed mutimode states.

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